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C^*-ALGEBRAS WITH HAUSDORFF SPECTRUM

BY

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ANSTRACT. By the spectrum of a C^*-algebra we mean the set of unitary equivalence classes of irreducible representations equipped with the hull-kernel topology. We are concerned with characterizing the C^*-algebras with identity which have Hausdorff spectrum. We characterize the C^*-algebras with identity and bounded representation dimension which have Hausdorff spectrum. Our results are more natural when the C^*-algebra is singly generated. For singly generated C^*-algebras with unbounded representation dimension, we reduce the problem to the case when the generator is an infinite direct sum of irreducible finite scalar matrices, and we have partial results in this case.

1. **Introduction and preliminaries.** For a C^*-algebra $\mathfrak{A}$ let $\hat{\mathfrak{A}}$ be the spectrum of $\mathfrak{A}$; that is, $\hat{\mathfrak{A}}$ is the set of unitary equivalence classes of nonzero irreducible representations of $\mathfrak{A}$ equipped with the hull-kernel topology [5, paragraph 3]. In this paper we attempt to characterize those C^*-algebras with identity that have Hausdorff spectrum. For A a bounded linear operator on a Hilbert space let $C^*(A)$ be the C^*-algebra generated by A and the identity. We say that A has Hausdorff spectrum if $C^*(A)\hat{}$ is Hausdorff. We started our research in this direction as a result of John Ernest's question of characterizing the operators A with Hausdorff spectrum [7]. Although we state many of our results for arbitrary separable C^*-algebras with identity, most of our results have more natural interpretations in the case of singly generated C^*-algebras.

It follows from J. Glimm's theorem [10, p. 582] that if $\mathfrak{A}$ is a separable C^*-algebra, then $\hat{\mathfrak{A}}$ is T_0 if and only if $\mathfrak{A}$ is GCR (or postliminal), and $\hat{\mathfrak{A}}$ is T_1 if and only if $\mathfrak{A}$ is CCR (or liminal). I. Kaplansky [13, Theorem 4.2] proved that if $\mathfrak{A}$ is a C^*-algebra such that all irreducible representations of $\mathfrak{A}$ are of the same finite dimension, then the primitive ideal space of $\mathfrak{A}$ is Hausdorff in the hull-kernel topology. In this case, the primitive ideal space is homeomorphic to $\hat{\mathfrak{A}}$, so that $\hat{\mathfrak{A}}$ is also Hausdorff. J. M. G. Fell proved a theorem [8, Corollary 1, p. 388] which has Kaplansky's result as a corollary.

We recall one characterization of Hausdorff spectrum that is in the literature

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[12], [18]. If $\mathcal{A}$ is a C^*-algebra with identity, and center $\mathcal{C}$, then $\mathcal{A}$ is called central if for all primitive ideals $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{A}$, $\mathcal{I} \cap \mathcal{C} = \mathcal{J} \cap \mathcal{C}$ implies that $\mathcal{I} = \mathcal{J}$. It follows from [12], [18] that if $\mathcal{A}$ is a separable C^*-algebra with identity, then $\hat{\mathcal{A}}$ is Hausdorff if and only if $\mathcal{A}$ is central and GCR. However we know of no natural way to compute the center of a singly generated C^*-algebra $C^*(A)$ in terms of the operator A, so we do not regard this necessary and sufficient condition as a satisfactory answer to the problem. We will make no use of this condition in our paper.

We say that a C^*-algebra $\mathcal{A}$ has bounded representation dimension if there is an integer N such that every irreducible representation of $\mathcal{A}$ acts on a Hilbert space of dimension less than or equal to N. In §2 we prove decomposition theorems, 2, 6, and 10, which give necessary and sufficient conditions for C^*-algebras with identity and bounded representation dimension to have Hausdorff spectrum. In §3 we show that, to characterize which operators A have Hausdorff spectrum, it suffices to consider only operators A which are (possibly infinite) direct sums of irreducible finite complex matrices. We are not able to give a complete characterization in the case of unbounded representation dimension, but we do give some partial results. We conclude with some remarks concerning the lifting of matrix units. We show that matrix units cannot necessarily be lifted from the Calkin algebra.

We recall the hull-kernel topology on $\hat{\mathcal{A}}$. For $S \subseteq \hat{\mathcal{A}}$, the closure of S is the set $\{\pi \in \hat{\mathcal{A}}: \text{Ker } \pi \supseteq \bigcap \{\text{Ker } \rho: \rho \in S\}\}$. The open sets of $\hat{\mathcal{A}}$ are all of the form $\{\pi \in \hat{\mathcal{A}}: \pi|\mathcal{J} \neq 0\}$ where $\mathcal{J}$ is a closed two-sided ideal in $\mathcal{A}$. If $\{A_i\}$ is a dense subset of $\mathcal{A}$, then the sets $Z_i = \{\pi \in \hat{\mathcal{A}}: \|\pi(A_i)\| > 1\}$ form a base for the topology of $\hat{\mathcal{A}}$. The tools used in the present paper are standard and are mostly contained in [5]. We will make frequent and extensive use of [5, §3.3], which was mostly taken from Fell's paper [8, §9].

We make some remarks on notation. We will use script letters $\mathcal{A}, \mathcal{B}, \ldots$ for C^*-algebras and Latin letters $A, B, \ldots$ for operators on a Hilbert space. We will denote the algebra of all bounded operators on a Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$ and the ideal of all compact operators by $\mathcal{K}(\mathcal{H})$. We will use $\mathcal{H}_\pi$ to denote the Hilbert space associated with some representation π: $\mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}_\pi)$. We denote $\{A \in \mathcal{A}: \pi(A) = 0\}$ by Ker π.

At the beginning of §2 we state our results both for C^*-algebras $\mathcal{A}$ and singly generated C^*-algebras $C^*(A)$. However, in the later parts of the paper we state our results only for algebras or operators.

2. **Some decomposition theorems and the case of bounded representation dimension.** Let $\mathcal{A}$ be a separable C^*-algebra with identity I and assume that $\hat{\mathcal{A}}$ is Hausdorff. If ρ and θ are irreducible representations of $\mathcal{A}$ and Ker ρ = Ker θ,

then ρ is in the closure of the singleton set $\{\theta\}$ in $\hat{A}$. Thus ρ and θ must be unitarily equivalent since $\hat{A}$ is Hausdorff. Hence, by [5, 9.1], A is GCR and $\rho(A)$ must contain $K(H_\rho)$. If $K(H_\rho)$ were properly contained in $\rho(A)$, then there would exist an irreducible representation π of A with Ker ρ properly contained in Ker π. This would contradict $\hat{A}$ being Hausdorff, hence we must have $\rho(A) = K(H_\rho)$, and $K(H_\rho)$ must have an identity. Thus H_ρ must be finite dimensional. We have thus proved the following theorem.

THEOREM 1. *If A is a separable C^*-algebra with identity such that $\hat{A}$ is Hausdorff, then every irreducible representation of A must be finite dimensional.*

We note that Theorem 1 is not true if we drop the hypothesis of A containing an identity. For if H is a separable infinite dimensional Hilbert space, then $K(H)$ is a separable C^*-algebra whose spectrum consists of a single point, and thus is Hausdorff, but has no representation of finite dimension. Theorem 1 is basic to much of what follows. The problem of characterizing when $\hat{A}$ is Hausdorff for algebras without identity seems more difficult, and in this paper we will usually assume A has an identity.

As in [15] we call a C^*-algebra A *n-normal* if for all $A_1, A_2, \ldots, A_{2n}$ in A we have

$$(*) \qquad \sum \operatorname{sgn}(\sigma) A_{\sigma(1)} A_{\sigma(2)} \ldots A_{\sigma(2n)} = 0$$

where the summation is taken over all permutations of $2n$ objects. An operator A is called n-normal if $C^*(A)$ is n-normal. We note that the identity $(*)$ is satisfied by the full $n \times n$ matrix algebra over a commutative C^*-algebra [15, §3]. A representation π of a C^*-algebra A is called n-normal if the C^*-algebra $\pi(A)$ is n-normal. An operator A is called *pure n-normal* if A is n-normal but no direct summand of A is k-normal for any $k < n$. A representation is called pure n-normal if it is n-normal but no subrepresentation is k-normal for any $k < n$.

THEOREM 2. *Let A be a separable C^*-algebra such that every irreducible representation of A is finite dimensional. If $\pi\colon A \to B(H_\pi)$ is a nondegenerate representation of A on a separable Hilbert space, then $\pi \cong \Sigma_{k\in I} \oplus \pi_k$ where each π_k is pure k-normal. If π is an n-normal representation then each $k \in I$ is less than or equal to n, and one need not assume that A or H_π are separable.*

PROOF. Since A is GCR we may apply Theorem 8.6.6 of [5] to obtain measures $\mu_1, \mu_2, \ldots, \mu_\infty$ on $\hat{A}$, pairwise disjoint, such that

$$(1) \qquad \pi \cong \int^{\oplus} \rho \, d\mu_1(\rho) \oplus 2\int^{\oplus} \rho \, d\mu_2(\rho) \oplus \ldots \oplus \aleph_0 \int^{\oplus} \rho \, d\mu_\infty(\rho).$$

If $\hat{A}_n = \{\theta \in \hat{A}: \dim(H_\theta) \leq n\}$, then $\hat{A} = \bigcup_{1 \leq n} \hat{A}_n$ and each $\hat{A}_n$ is closed in A [5, 3.6.3]. We may thus write, for each i and n,

$$\int^{\oplus} \rho \, d\mu_i(\rho) = \int^{\oplus}_{\hat{A}_n} \rho \, d\mu_i(\rho) \oplus \int^{\oplus}_{X_n} \rho \, d\mu_i(\rho) \tag{2}$$

where X_n is the complement of $\hat{A}_n$ in $\hat{A}$. Now for some n and i we must have that $\mu_i(\hat{A}_n) \neq 0$, in which case the first representation in the right-hand side of equation (2) is nondegenerate and n-normal. By applying the decomposition of (2) to (1) and rearranging, we have for some n that $\pi \cong \theta \oplus \theta'$, where θ is an n-normal nondegenerate representation. We have thus shown that if π is any nondegenerate representation of A on a separable Hilbert space, then some subrepresentation of π is n-normal for some n. Now let n_0 be the smallest integer such that π has a subrepresentation that is n_0-normal. By Zorn's lemma there is a maximal family $F_{n_0} = \{H_\alpha^{(n_0)}\}$ of orthogonal subspaces of H_π, each reducing π, such that each $\pi | H_\alpha^{(n_0)}$ is n_0-normal. Let $H^{(n_0)} = \Sigma_\alpha \oplus H_\alpha^{(n_0)}$. Now pick a maximal family $F_{n_0+1} = \{H_\beta^{(n_0+1)}\} \supset F_{n_0}$ of orthogonal subspaces of H_π, each reducing π, such that each $\pi | H_\beta^{(n_0+1)}$ is $(n_0 + 1)$-normal. Let $H^{(n_0+1)} = \Sigma_\beta \oplus H_\beta^{(n_0+1)}$. Continue to choose maximal families $F_{n+1} = \{H_\gamma^{(n+1)}\} \supseteq F_n$ of orthogonal subspaces of H_π, each reducing π, such that $\pi | H_\gamma^{(n+1)}$ is $(n + 1)$-normal. Again let $H^{(n+1)} = \Sigma_\gamma \oplus H_\gamma^{(n+1)}$. Then $H^{(n_0)} \subseteq H^{(n_0+1)} \subseteq \ldots$, and each $\pi | H^{(n)}$ is n-normal. Now let $H_{n_0} = H^{(n_0)}$ and $H_{n+1} = H^{(n+1)} \ominus H^{(n)}$ for $n \geq n_0$. Let $I = \{k \in Z^+: H_k \neq \{0\}\}$, and let $H_\infty = \Sigma_{k \in I} \oplus H_k$. Since $\pi | H \ominus H_\infty$ is a subrepresentation of π which by maximality contains no k-normal subrepresentation for any k, we obtain that $H = H_\infty$. Furthermore, for $k \in I$, $\pi_k = \pi | H_k$ is clearly k-normal, and must be pure k-normal by the maximality of F_{k-1}. Hence $\pi = \Sigma_{k \in I} \oplus \pi_k$ with each π_k pure k-normal. If π is a nondegenerate n-normal representation of any C^*-algebra A then the first part of the proof is superfluous since any subrepresentation of π is n-normal. In this case $H = H^{(n)}$ and we must have that $k \in I$ implies $k \leq n$.

COROLLARY 3. *If A is a bounded operator on a separable Hilbert space such that every irreducible representation of $C^*(A)$ is finite dimensional, then $A \cong \Sigma_{k \in I} \oplus A_k$ where each A_k is pure k-normal.*

PROOF. Apply Theorem 2 to the identity representation of $C^*(A)$ and let $A_i = \pi_i(A)$.

Let A be a C^*-algebra and let $\theta: A \longrightarrow B(H)$ be a representation. Then the map $\hat{\theta}: \theta(A)\hat{} \longrightarrow \hat{A}$ defined by $\hat{\theta}(\rho) = \rho \circ \theta$ is one-to-one and continuous. Thus $\theta(A)\hat{}$ must be Hausdorff if $\hat{A}$ is Hausdorff. Hence if A is a separable C^*-algebra with identity and Hausdorff spectrum, then the C^*-algebras $\pi_i(A)$

in Theorem 2 also have Hausdorff spectrum. Likewise, if A has Hausdorff spectrum, then the A_i in Corollary 3 also have Hausdorff spectrum. We need the following two lemmas which will be used several times.

LEMMA 4. *Let* θ: $\mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ *be a representation of a* C^**-algebra* $\mathcal{A}$. *If* $\theta(\mathcal{A})\hat{}$ *is Hausdorff, then* $\hat{\theta}(\theta(\mathcal{A})\hat{})$ *is closed in* $\mathcal{A}\hat{}$ *and is a Hausdorff space in the relative topology from* $\mathcal{A}\hat{}$.

PROOF. This follows immediately from [5, 3.2.1].

LEMMA 5. *Let* π: $\mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ *be a representation of a* C^**-algebra* $\mathcal{A}$ *such that* $\pi = \Sigma_{i \in I} \oplus \pi_i$, *each* π_i *a representation of* $\mathcal{A}$. *Then each* π_i *gives rise to a representation of* $\pi(\mathcal{A})$ *into* $\pi_i(\mathcal{A})$, *which we call* P_i, *defined by* $P_i(\pi(A)) = \pi_i(A)$. *Then* $\bigcup_{i \in I} \hat{P}_i(\pi_i(\mathcal{A})\hat{})$ *is dense in* $\pi(\mathcal{A})\hat{}$.

PROOF. Let $U = \{\theta \in \pi(\mathcal{A})\hat{}: \theta | \mathcal{J} \neq 0\}$ be any nonempty open set in $\pi(\mathcal{A})\hat{}$, where $\mathcal{J}$ is a nonzero closed ideal in $\pi(\mathcal{A})$. Let $0 \neq \pi(A) \in \mathcal{J}$ and choose i with $\pi_i(A) \neq 0$. Then for $\rho_i \in \pi_i(\mathcal{A})\hat{}$ with $\rho_i(\pi_i(A)) \neq 0$ we have that $\hat{P}_i(\rho_i) \in U$. Hence $\bigcup_{i \in I} \hat{P}_i(\pi_i(\mathcal{A})\hat{})$ is dense in $\pi(\mathcal{A})\hat{}$.

The following theorem is now immediate from Lemmas 4 and 5.

THEOREM 6. *Let* π *be a nondegenerate representation of a separable* C^**-algebra* $\mathcal{A}$ *with identity such that* $\pi = \Sigma_{i=1}^n \oplus \pi_i$, n *a finite integer, and each* π_i *a nondegenerate representation of* $\mathcal{A}$. *Then* $\pi(\mathcal{A})$ *has Hausdorff spectrum if and only if each* $\pi_i(\mathcal{A})$ *has Hausdorff spectrum. If every irreducible representation of* $\mathcal{A}$ *is finite dimensional, then we do not need to assume that* $\mathcal{A}$ *is separable.*

PROOF. Since we have representations P_i: $\pi(\mathcal{A}) \longrightarrow \pi_i(\mathcal{A})$, each $\pi_i(\mathcal{A})$ has Hausdorff spectrum if $\pi(\mathcal{A})$ has Hausdorff spectrum. On the other hand if each $\pi_i(\mathcal{A})$ has Hausdorff spectrum, then by Lemmas 4 and 5 we have that $\pi(\mathcal{A})\hat{} = \bigcup_{i=1}^n \hat{P}_i(\pi_i(\mathcal{A})\hat{})$ and each $\hat{P}_i(\pi_i(\mathcal{A})\hat{})$ is closed in $\pi(\mathcal{A})\hat{}$ and is a Hausdorff space in the relative topology from $\pi(\mathcal{A})\hat{}$. It is then clear that every net in $\pi(\mathcal{A})\hat{}$ has a unique limit point.

COROLLARY 7. *If* A *is a bounded operator on a Hilbert space and* A *is a finite direct sum of operators* A_i, $A = \Sigma_{i=1}^n \oplus A_i$, *then* A *has Hausdorff spectrum if and only if each* A_i *has Hausdorff spectrum.*

We will give an example in §3 to show that if $\pi = \Sigma_{i=1}^\infty \oplus \pi_i$, then each $\pi_i(\mathcal{A})\hat{}$ can be Hausdorff without $\pi(\mathcal{A})\hat{}$ being Hausdorff; thus the hypothesis of only a finite number of direct summands in Theorem 6 and Corollary 7 is necessary. Theorem 6 implies the following (probably known) corollary.

COROLLARY 8. *Every finite dimensional C^*-algebra has Hausdorff spectrum.*

PROOF. First, we include a proof that a finite dimensional C^*-algebra A has only a finite number of unitarily inequivalent irreducible representations. Let $\rho_1, \rho_2, \ldots, \rho_n$ be unitarily inequivalent irreducible representations of A and let $\rho = \Sigma_{i=1}^n \oplus \rho_i$, $\rho: A \longrightarrow B(H_\rho)$. By Kadison's transitivity theorem [5, 2.8.3], ρ is a cyclic representation. Hence $n \leq \dim H_\rho \leq \dim A$. Now let $\rho_1, \rho_2, \ldots, \rho_k$ be all the irreducible representations of A up to unitary equivalence, and let $\rho = \Sigma_{i=1}^k \oplus \rho_i$. Then A is isomorphic to $\rho(A)$. Since each $\rho_i(A)$ is a full finite matrix algebra, each $\rho_i(A)\hat{}$ is a single point and thus Hausdorff. Theorem 6 then implies that $\rho(A)\hat{}$ and hence $\hat{A}$ is Hausdorff.

If A has bounded representation dimension N, then A is N-normal. For A a C^*-algebra with identity and bounded representation dimension, let π be a faithful nondegenerate representation of A. Then by Theorem 2 we can write π as a finite direct sum of pure k-normal representation. Theorem 6 then implies that in order to characterize when a C^*-algebra with identity and bounded representation dimension has Hausdorff spectrum, one need only characterize when $\pi(A)$ has Hausdorff spectrum for π a pure k-normal representation of A. This is done in Theorem 10.

Let $\pi: A \longrightarrow B(H)$ be an n-normal representation. Then, by [15], $\pi(A)'' \cong \Sigma_{k=1}^n \oplus M_k(C_k)$, where $M_k(C_k)$ is the algebra of $k \times k$ matrices with entries from the abelian W^*-algebra C_k. Let I_i be the element in $\Sigma_{k=1}^n \oplus M_k(C_k)$ with the identity in the ith coordinate and zeros elsewhere. Then $\pi(A) \longrightarrow I_i\pi(A)I_i$ is an i-normal subrepresentation of π. Thus if π is a pure n-normal representation then $\pi(A)'' \cong M_n(C_n)$. In this case, let $\mathcal{D}$ be the C^*-subalgebra of C_n generated by the matrix entries from elements of $\pi(A)$. Let $X(\pi)$ be the maximal ideal space of $\mathcal{D}$. For $\rho \in X(\pi)$, let $\hat{\rho}$ be defined on $M_n(\mathcal{D})$ by $\hat{\rho}((D_{ij})) = (\rho(D_{ij}))$. Then $\hat{\rho}$ is a representation of $M_n(\mathcal{D})$ into $B(\mathbf{C}^n)$. As in [2, Proposition 2] every irreducible representation of $\pi(A)$ is of the form $\pi(A) \longrightarrow \hat{\rho}(\pi(A))|M$ for some $\rho \in X(\pi)$ and for some subspace M reducing for $\hat{\rho}(\pi(A))$.

LEMMA 9. *Let $\pi: A \longrightarrow B(H)$ be a pure n-normal representation of a C^*-algebra A with identity. Then $Y \equiv \{\omega \in X(\pi): \hat{\omega}|\pi(A)$ is irreducible$\}$ is dense in $X(\pi)$.*

PROOF. Let $\mathcal{D}$ and $X(\pi)$ be defined as above. Then $\mathcal{D} \cong C(X(\pi))$ the set of all continuous complex-valued functions on $X(\pi)$. Suppose that $\theta: C(X(\pi)) \longrightarrow B(H')$ is a faithful representation of $C(X(\pi))$ on a Hilbert space H', and let E be the regular spectral measure associated with θ by the general spectral theorem. That is, $\theta(f) = \int f dE$ for all $f \in C(X(\pi))$. Let $\hat{\theta}$ be the associated representation of $M_n(C(X(\pi)))$ on the direct sum of n-copies of H', defined by $\hat{\theta}((f_{ij})) = (\theta(f_{ij}))$. Then π is unitarily equivalent to the representation $A \mapsto \hat{\theta}(\pi(A))$, where we identify $\pi(A)$ as an element of $M_n(C(X(\pi)))$. Now let S be the complement of $\overline{Y}$ in $X(\pi)$. Define the representation σ: $C(X(\pi)) \longrightarrow B(E(S)H')$ by $\sigma(f) = \theta(f)|E(S)H'$, and let $\hat{\sigma}$ be the associated representation of $M_n(C(X(\pi)))$ on the direct sum of n-copies of $E(S)H'$. Now, for $\omega \in S$, $\hat{\omega}(\pi(A))$ is $(n-1)$-normal, and a computation then shows that $\hat{\sigma}(\pi(A))$ is also $(n-1)$-normal. But the representation $A \mapsto \hat{\sigma}(\pi(A))$ is a subrepresentation of the representation $A \mapsto \hat{\theta}(\pi(A))$. Since π, and hence $A \mapsto \hat{\theta}(\pi(A))$, is pure n-normal we must have that $E(S) = 0$. Hence support$(E) \subset \overline{Y}$. But θ is faithful, so if $f \in C(X(\pi))$ satisfies f(support $E) = 0$ then $f = 0$. Hence $\overline{Y} = X(\pi)$ and the lemma is proved.

THEOREM 10. *Let π: $A \longrightarrow B(H)$ be a pure n-normal representation of a C^*-algebra A with identity. Then $\pi(A)$ has Hausdorff spectrum if and only if, for every $\rho \in X(\pi)$, $\hat{\rho}|\pi(A)$ is a direct sum of unitarily equivalent irreducible representations of $\pi(A)$.*

PROOF. First assume that, for all $\rho \in X(\pi)$, $\hat{\rho}|\pi(A)$ is a direct sum of unitarily equivalent irreducible representations of $\pi(A)$. Let π_α be a net in $\pi(A)\hat{}$ converging to both π_1 and π_2 in $\pi(A)\hat{}$. Again applying [2, Proposition 2] we obtain that for each α there is a $\omega_\alpha \in X(\pi)$ and a reducing subspace M_α for $\hat{\omega}_\alpha(\pi(A))$ such that $\pi_\alpha(\pi(A)) = \hat{\omega}_\alpha(\pi(A))|M_\alpha$. Since $X(\pi)$ is compact we can assume by passing to a subset that ω_α converges to $\omega_0 \in X(\pi)$. Now consider a neighborhood of π_1 of the form $U = \{\theta \in \pi(A)\hat{}: \|\theta(\pi(A))\| > 1\}$, where A is a fixed but arbitrary element of A. Let $\epsilon > 0$ be such that $\|\pi_1(\pi(A))\| > 1 + \epsilon$. Then there is an α_0 such that $\|\pi_\alpha(\pi(A))\| > 1 + \epsilon$ for all $\alpha \geqslant \alpha_0$. Since ω_α converges to ω_0, we have that $\hat{\omega}_\alpha(\pi(A))$ converges to $\hat{\omega}_0(\pi(A))$ in norm, for all $A \in A$. But, by our assumption, $\hat{\omega}_0|\pi(A) \cong \Sigma_{i=1}^k \oplus \pi_0$ for some irreducible representation π_0 of $\pi(A)$, and $\hat{\omega}_\alpha|\pi(A) \cong \Sigma_{i=1}^k \oplus \pi_\alpha$ (this fact follows from our assumption and [5, 5.3]). Then $\|\hat{\omega}_0(\pi(A))\| = \|\pi_0(\pi(A))\|$ and $\|\hat{\omega}_\alpha(\pi(A))\| = \|\pi_\alpha(\pi(A))\|$ for all α. Hence $\|\pi_\alpha(\pi(A))\|$ converges to $\|\pi_0(\pi(A))\|$ and we obtain that $\|\pi_0(\pi(A))\| \geqslant 1 + \epsilon$. Hence $\pi_0 \in U$ and $\{\pi_0\}$ is in the closure of the singleton set $\{\pi_1\}$. But since $\pi(A)$ is a CCR algebra and CCR algebras have T_1 spectrum [5, 4.1.10 and 4.1.11], this implies that $\pi_0 \cong \pi_1$. Likewise $\pi_0 \cong \pi_2$ and $\pi(A)\hat{}$ is Hausdorff.

Now assume that there is a $\omega_0 \in X(\pi)$ such that $\hat{\omega}_0|\pi(A) \cong \pi_1 \oplus \pi_2 \oplus \pi'$ where π_1 and π_2 are unitarily inequivalent irreducible representations of $\pi(A)$. Now by Lemma 9 there exists a net $\omega_\alpha \in X(\pi)$ such that ω_α converges to ω_0 and $\hat{\omega}|\pi(A)$ is irreducible. Let $U = \{\theta \in \pi(A)\hat{}: \|\theta(\pi(A))\| > 1\}$ be an open set containing π_1. Since $\|\hat{\omega}_\alpha(\pi(A))\|$ converges to $\|\hat{\omega}_0(\pi(A))\|$ which is greater than or equal to $\|\pi_1(\pi(A))\|$, we have that $\hat{\omega}_\alpha|\pi(A) \in U$ for all $\alpha \geqslant \alpha_0$ for some α_0. Hence $\hat{\omega}_\alpha|\pi(A)$ converges to π_1, and likewise to π_2. Thus $\pi(A)\hat{}$ is not Hausdorff.

Theorem 2, Theorem 6, and Theorem 10 together give concrete necessary and sufficient conditions for a C^*-algebra with identity and bounded representation dimension to have Hausdorff spectrum. This result includes Kaplansky's result [13, Theorem 4.2] in the case when the algebra has an identity.

With the aid of Theorem 10 we now give some simple examples. Let M_t be defined on $L^2(0, 1)$ by $(M_t f)(x) = xf(x)$ for all $f \in L^2(0, 1)$. For $\alpha, \beta \in \mathbf{C}$, let $A_{\alpha,\beta}$ be the operator matrix

$$\begin{bmatrix} \alpha I & M_t \\ 0 & \beta I \end{bmatrix}$$

defined on $L^2(0, 1) \oplus L^2(0, 1)$. It is easily seen that $A_{\alpha,\beta}$ is always pure 2-normal. Theorem 10 implies that $A_{\alpha,\beta}$ has Hausdorff spectrum if and only if $\alpha = \beta$. We remark that this result also follows easily from the results in [2]. Thus the operator $A_{1,1}$ is an example of a nonnormal operator with Hausdorff spectrum which has irreducible representations of dimensions one and two, while the operator $A_{1,-1}$ is such an example with non-Hausdorff spectrum. Thus $C^*(A_{1,-1})$ is a C^*-algebra whose spectrum is non-Hausdorff, but which is a C^*-subalgebra of $M_2(C[0, 1])$ whose spectrum is Hausdorff. Thus the property of having Hausdorff spectrum is not inherited by subalgebras, as is the property of separable C^*-algebras having T_0 or T_1 spectrum [5, 4.2.4 and 4.3.5].

3. **The case of unbounded representation dimension.** In this section we deal only with singly generated C^*-algebras. Let A be a bounded operator on a separable Hilbert space and assume that every irreducible representation of $C^*(A)$ is finite dimensional. Then by Corollary 3 we can write $A = \Sigma_{k\in I} \oplus A_k$ where each A_k is pure k-normal. Theorem 11 will show that in order to determine when $C^*(A)$ has Hausdorff spectrum, it suffices to solve the case when each A_k is actually an irreducible finite complex matrix.

THEOREM 11. *Let A be a bounded operator on a Hilbert space and assume that A is the direct sum of operators A_k, $A = \Sigma_{k=1}^\infty \oplus A_k$. Then A has Hausdorff spectrum if and only if each A_k has Hausdorff spectrum and for all choices of $\theta_k \in C^*(A_k)\hat{}$ the operator $\Sigma_{k=1}^\infty \oplus \theta_k(A_k)$ has Hausdorff spectrum.*

PROOF. Assume that A has Hausdorff spectrum. Then for every k the operator A_k has Hausdorff spectrum since the mapping $A \longrightarrow A_k$ is a representation of $C^*(A)$ (recall that such a representation induces a continuous one-to-one mapping from $C^*(A_k)^\wedge$ to $C^*(A)^\wedge$). Likewise for all choices $\theta_k \in C^*(A_k)^\wedge$ the mapping $A \longrightarrow \Sigma_{k=1}^\infty \oplus \theta_k(A_k)$ is a representation of $C^*(A)$ so that $\Sigma_{k=1}^\infty \oplus \theta_k(A_k)$ has Hausdorff spectrum.

Now assume that each A_i has Hausdorff spectrum and that $\Sigma_{k=1}^\infty \oplus \theta_k(A_k)$ has Hausdorff spectrum for all choices $\theta_k \in C^*(A_k)^\wedge$. Furthermore suppose there exist $\rho_0, \rho_1 \in C^*(A)^\wedge$ which cannot be separated by open sets. Let $\{U_i\}$ and $\{V_j\}$ be countable bases for the open sets containing ρ_0 and ρ_1 respectively. Since ρ_0 and ρ_1 cannot be separated by open sets we must have that $U_j \cap V_j \neq \emptyset$ for all j. By Lemma 5 for each j we can choose an element $\varphi_j \in U_j \cap V_j$ with $\varphi_j \in \bigcup_{k=1} \hat{P}_k(C^*(A_k)^\wedge)$ where $P_k(A) = A_k$. Then the sequence φ_j converges to both ρ_0 and ρ_1. Since each A_k has Hausdorff spectrum, Lemma 4 implies that each $\hat{P}_k(C^*(A_k)^\wedge)$ is closed in $C^*(A)^\wedge$ and is a Hausdorff space in the relative topology from $C^*(A)^\wedge$. Hence only a finite number of the $\{\varphi_j\}$ belong to any one $\hat{P}_k(C^*(A_k)^\wedge)$, and by passing to a subsequence we may assume that if $i \neq j$ then φ_i and φ_j belong to different $\hat{P}_k(C^*(A_k)^\wedge)$. Thus for every j there exists k_j such that $\varphi_j \in \hat{P}_{k_j}(C^*(A_{k_j})^\wedge)$ and $i \neq j$ implies $k_i \neq k_j$. Now since $\varphi_j \in \hat{P}_{k_j}(C^*(A_{k_j})^\wedge)$ there is a $\theta_{k_j} \in C^*(A_{k_j})^\wedge$ such that $\varphi_j(A) = \theta_{k_j}(P_{k_j}(A)) = \theta_{k_j}(A_{k_j})$. Now let θ be a representation of $C^*(A)$ defined by $\theta(A) = \Sigma_{j=1}^\infty \oplus \theta_{k_j}(A_{k_j})$. Since, by Corollary 7, a direct summand of an operator with Hausdorff spectrum has Hausdorff spectrum, $\theta(A)$ has Hausdorff spectrum. Define representations $\widetilde{\theta}_{k_j}, \widetilde{\rho}_0, \widetilde{\rho}_1$ on $C^*(\theta(A))$ by

$$\widetilde{\theta}_{k_j}(p(\theta(A), \theta(A)^*)) = \theta_{k_j}(p(A_{k_j}, A_{k_j}^*))$$

and

$$\widetilde{\rho}_i(p(\theta(A), \theta(A)^*)) = \rho_i(p(A, A^*)),$$

for p a polynomial in two noncommuting variables. It is clear that θ_{k_j} extends to an irreducible representation of $C^*(\theta(A))$. Since φ_j converges to both ρ_0 and ρ_1 in $C^*(A)^\wedge$, lower semicontinuity of the norm [4, 3.3.2] gives

$$\begin{aligned}\|\rho_i(p(A, A^*))\| &\leqslant \liminf \|\varphi_j(p(A, A^*))\| \\ &\leqslant \sup \|\theta_{k_j}(p(A_{k_j}, A_{k_j}^*))\| = \|\theta(p(A, A^*))\|.\end{aligned}$$

So that $\widetilde{\rho}_i$ also extend to irreducible representations of $C^*(\theta(A))$. Also the same computation shows that $\widetilde{\theta}_{k_j}$ converges to both $\widetilde{\rho}_0$ and $\widetilde{\rho}_1$ in $C^*(\theta(A))^\wedge$. Hence $\widetilde{\rho}_0 \cong \widetilde{\rho}_1$, which implies $\rho_0 \cong \rho_1$. Contradiction. Hence $C^*(A)^\wedge$ must be Hausdorff.

As previously mentioned, if A is a bounded operator on a separable Hilbert space such that every irreducible representation of $C^*(A)$ is finite dimensional, then by means of Corollary 3, Theorem 6, Theorem 10, and Theorem 11 one could decide if $C^*(A)^\wedge$ had Hausdorff spectrum if one could settle the question for operators of the form $\Sigma_{k\in I} \oplus A_k$ with each A_k an irreducible $k \times k$ complex matrix. At the present time we are unable to resolve this question, but we do present some partial results.

Let B be any operator on a Hilbert space of dimension N, and let I_n denote the identity operator on a Hilbert space of dimension n. Since the set of irreducible operators on any separable Hilbert space is dense [11, p. 920], for every n there is an operator K_n with $\|K_n\| < 1/n$ such that $A_n = B \otimes I_n + K_n$ is an irreducible operator on the Hilbert space of dimension nN. Let $A = \Sigma_{n=2}^{\infty} \oplus A_n$. Then the following theorem completely describes $C^*(A)^\wedge$ and its topology in this case.

THEOREM 12. *Let A be as above and let $B_1, B_2, \ldots, B_k$ be all the unitarily inequivalent irreducible direct summands of B. Then for every $1 \leq i \leq k$ there is an irreducible representation π_i of $C^*(A)$ determined by $\pi_i(A) = B_i$; and $C^*(A)^\wedge = \{\pi_i: 1 \leq i \leq k\} \cup \{P_n: 2 \leq n\}$, where as before $P_n(A) = A_n$. The topology is determined by the fact that singleton sets are closed and the sequence $\{P_n\}_{n=2}^{\infty}$ converges to each π_i. Thus A has Hausdorff spectrum if and only if B is a direct sum of unitarily equivalent irreducible matrices.*

PROOF. For any polynomial p in two noncommuting variables we have that

$$\|p(B, B^*)\| = \|p(B \otimes I_n, B^* \otimes I_n)\| = \|p(A_n, A_n^*) + K_n'\|$$

where $\|K_n'\|$ converges to zero as n tends to infinity. Hence

$$\begin{aligned}\|p(B, B^*)\| &\leq \limsup(\|p(A_n, A_n^*)\| + \|K_n'\|) \\ &\leq \sup\|p(A_n, A_n^*)\| = \|p(A, A^*)\|.\end{aligned}$$

Hence there is a representation π of $C^*(A)$ determined by $\pi(A) = B$, and then there are irreducible representations π_i of $C^*(A)$ determined by $\pi_i(A) = B_i$. Now, for any polynomial p and for all i and n,

$$\begin{aligned}\|\pi_i(p(A, A^*))\| &= \|p(B_i, B_i^*)\| \leq \|p(B, B^*)\| \\ &= \|p(A_n, A_n^*) + K_n'\| \leq \|p(A_n, A_n^*)\| + \|K_n'\| \\ &= \|P_n(p(A, A^*))\| + \|K_n'\|.\end{aligned}$$

Hence if $\|\pi_i(p(A, A^*))\| > 1$ then there is an n_0 such that $\|P_n(p(A, A^*))\| > 1$ for all $n \geq n_0$. Since the algebra of polynomials in A and A^* is dense in $C^*(A)$

we obtain that the sequence $\{P_n\}_{n=2}^{\infty}$ converges to each π_i.

We now show that the $\{\pi_i\}$ and $\{P_n\}$ are the only irreducible representations of $C^*(A)$ up to unitary equivalence. Let $\theta\colon C^*(A) \longrightarrow \mathcal{B}(H_\theta)$ be an irreducible representation of $C^*(A)$. By [5, 2.10.2], we can "extend" θ to an irreducible representation $\theta'\colon \Sigma_{n=2}^{\infty} \oplus C^*(A_n) \longrightarrow \mathcal{B}(H_{\theta'})$, where H_θ is a subspace of $H_{\theta'}$ reducing $\theta'(A)$ and $\theta'(A)|H_\theta = \theta(A)$. Here $\Sigma_{n=2}^{\infty} \oplus C^*(A_n)$ is the C^*-algebra of all bounded sequences with entries from the $C^*(A_n)$. Let I_j be the operator in $\Sigma_{n=2}^{\infty} \oplus C^*(A_n)$ with I in the jth coordinate and zeros elsewhere. Since θ' is irreducible and since $\theta'(I_j)$ is a projection, $\theta'(I_j)$ is either zero or I and $\theta'(I_j) = I$ for at most one j. Suppose $\theta'(I_j) = I$. Then $\theta'(AI_j) = \theta'(A)$, and we have an irreducible representation of $C^*(A_j)$, determined by $A_j \longrightarrow \theta'(AI_j)|H_\theta = \theta'(A)|H_\theta = \theta(A)$. But since A_j is a finite irreducible matrix this implies that $\theta(A) \cong A_j$ and $\theta \cong P_j$. Now suppose $\theta'(I_j) = 0$ for all j. Then $\theta'(\Sigma_{n=2}^{\infty} \oplus K_n) = 0$ so that $\theta'(\Sigma_{n=2}^{\infty} \oplus (B \otimes I_n)) = \theta'(\Sigma_{n=2}^{\infty} \oplus A_n) = \theta'(A)$. Hence one has an irreducible representation of $C^*(B)$ determined by $B \longrightarrow \theta'(\Sigma_{n=1}^{\infty} \oplus (B \otimes I_n))|H_\theta = \theta'(A)|H_\theta = \theta(A)$. Hence $\theta(A) \cong B_i$ for some i and thus $\theta \cong \pi_i$ for some i. Thus $C^*(A)\hat{} = \{\pi_i\colon 1 \leqslant i \leqslant k\} \cup \{P_n\colon 2 \leqslant n\}$ and all the points are distinct; notice that $\dim(\pi_i) \leqslant N < \dim(P_n)$ for all $1 \leqslant i \leqslant k$ and $2 \leqslant n$. Since all irreducible representations of $C^*(A)$ are finite dimensional, $C^*(A)$ is CCR and hence $C^*(A)\hat{}$ is T_1 [5, 4.1.10 and 4.1.11], so that singleton sets are closed. The topology is then completely determined by the fact that the sequence $\{P_n\}$ converges to each π_i. For then each P_n is open and for each i the sets $\{P_n\colon m \leqslant n\} \cup \{\pi_i\}$ form a base for the open sets containing π_i. It follows that A has Hausdorff spectrum if and only if $k = 1$, that is, B is the direct sum of unitarily equivalent irreducible matrices.

Theorem 12 shows that Theorem 6 does not extend to the case of an infinite number of direct summands. Also, Theorem 12 shows that there exist operators A with Hausdorff spectrum such that $C^*(A)$ does not have bounded representation dimension.

Let A be an operator with the structure of the operator A in Theorem 12; that is, $A = \Sigma_{n=2}^{\infty} \oplus A_n$ with $A_n = B \otimes I_n + K_n$, A_n irreducible, $\|K_n\| \longrightarrow 0$, and B a finite matrix. Since the set $\{P_n\colon 2 \leqslant n\}$ is discrete in $C^*(A)\hat{}$, an application of the Dauns-Hofmann theorem [6, Remark 7] implies that $I_j \in C^*(A)$ for all j. Hence $C^*(A)$ contains the C^*-algebra $\Sigma' = \Sigma' \oplus C^*(A_n)$ of all sequences in $\Sigma \oplus C^*(A_n)$ which converge to zero in norm. This observation motivates our next considerations.

Suppose $A = \Sigma \oplus A_n$ with each A_n an irreducible finite matrix and that $\Sigma' \subseteq C^*(A)$. Furthermore, assume that the sequence $\{P_n\}_{n=2}^{\infty}$ converges to a unique irreducible representation π with $\pi(A) = B$ (thus $C^*(A)\hat{}$ consists of a discrete sequence with a single limit point and is hence Hausdorff, thus π must be finite dimensional). As a small step toward completing the characterization of operators with Hausdorff spectrum, we will show that there is an n_0 such that, for all $n \geqslant n_0$, $A_n \cong B \otimes I + K_n$ with $\|K_n\| \longrightarrow 0$, and I is the identity on an appropriate space (note that included in this result is the fact that the dimension of the space that π acts on must divide the dimension of the space that P_n acts on for all $n \geqslant n_0$). Thus we will obtain a concrete characterization of those operators $A = \Sigma_{n=1}^{\infty} \oplus A_n$, A_n an irreducible finite matrix, such that $C^*(A)\hat{}$ is a countable set with a single limit point. Since $C^*(A)\hat{} = \{P_n: 2 \leqslant n\} \cup \{\pi\}$, the C^*-algebra $C^*(A)/\Sigma'$ is isomorphic to $\mathcal{B}(\mathcal{H}_\pi)$, and we consider π as the quotient map of $C^*(A)$ onto $C^*(A)/\Sigma'$, and also as the quotient map of $\Sigma \oplus C^*(A_n)$ onto $(\Sigma \oplus C^*(A_n))/\Sigma'$.

If the dimension of π is one, then $\pi(A)$ is a scalar, say $\pi(A) = \lambda$. But then $\pi(A - \lambda I) = 0$ so $A - \lambda I \in \Sigma'$ and A has the desired structure. The case when π has higher dimension is somewhat harder. We need to show that matrix units in $C^*(A)/\Sigma'$ can be lifted to "almost matrix units" in $C^*(A)$. We first need some lemmas. Lemma 13 is known and was shown to us by the late David Topping. However, we do not know of a published reference, so we include a proof.

LEMMA 13. *Let E and F be projections in $\mathcal{B}(\mathcal{H})$. If $\|EF\| < 1$, then $E(\mathcal{H}) \cap F(\mathcal{H}) = \{0\}$ and $E(\mathcal{H}) + F(\mathcal{H})$ is closed. Furthermore, if P is defined by $Pz = 0$ for $z \in (E(\mathcal{H}) + F(\mathcal{H}))^\perp$ and $P(x + y) = x$ for $x \in E(\mathcal{H})$, $y \in F(\mathcal{H})$, then P is bounded and $\|P\| \leqslant (1 - \|EF\|)^{-1/2}$.*

PROOF. Actually, a more detailed analysis than we will do shows that $\|P\| = (1 - \|EF\|^2)^{-1/2}$. However, the estimate in the lemma is all that we will need. Clearly $E(\mathcal{H}) \cap F(\mathcal{H}) = 0$. If $Ex = x$ and $Fy = y$, then $|(x, y)| \leqslant \|EF\|\,\|x\|\,\|y\|$, so that

$$\begin{aligned}\|x + y\|^2 &\geqslant \|x\|^2 - 2\|EF\|\,\|x\|\,\|y\| + \|y\|^2 \\ &= (1 - \|EF\|)(\|x\|^2 + \|y\|^2) + \|EF\|(\|x\| - \|y\|)^2 \\ &\geqslant (1 - \|EF\|)\,\|x\|^2.\end{aligned}$$

It then follows that $E(\mathcal{H}) + F(\mathcal{H})$ is closed and $\|P\| \leqslant (1 - \|EF\|)^{-1/2}$.

LEMMA 14. *Let E and F be projections in $\mathcal{B}(\mathcal{H})$ and assume that $\|EF\| < 1$. Then*

$$\|(E \vee F) - E - F\| \leqslant 2\|EF\|(1 - \|EF\|)^{-1/2}.$$

Here $E \vee F$ is the supremum of the projections E and F.

PROOF. The right-hand side of the inequality can be improved to $(2\|EF\|)\cdot(1 - \|EF\|^2)^{-1/2}$, but the stated estimate is all that we need. Let P and Q in $B(H)$ be defined by $Pz = 0 = Q(z)$ if $z \in (E(H) + F(H))^\perp$ and, for $Ex = x$, $Fy = y$, let $P(x + y) = x$, $Q(x + y) = y$. Then for $z = x + y$ with $Ex = x$, $Fy = y$ we obtain that

$$\begin{aligned}\|(E \vee F - E - F)z\| &= \|x + y - x - Ey - Fx - y\| \\ &= \|Ey + Fx\| = \|EQz + FPz\| \\ &\leqslant \|EFQz + FEPz\| \leqslant \|EF\|(\|Q\| + \|P\|)\,\|z\| \\ &\leqslant 2\|EF\|(1 - \|EF\|)^{-1/2}\|z\|,\end{aligned}$$

where the last inequality is from Lemma 13. Hence Lemma 14 follows.

We now prove that one can lift a finite family of orthogonal projections from $C^*(A)/\Sigma'$ back to $C^*(A)$. It is known that orthogonal projections can be lifted out of the Calkin algebra [20, Lemma 3.4], but we want to get our projections in the C^*-algebra $C^*(A)$.

LEMMA 15. *Suppose that $A = \Sigma_{n=2}^\infty \oplus A_n$, with each A_n an irreducible finite matrix and $\Sigma' \subseteq C^*(A)$. If $\{e_i\colon 1 \leqslant i \leqslant N\}$ is a finite family of orthogonal projections in $C^*(A)/\Sigma'$, then there is a family of orthogonal projections $\{E_i\colon 1 \leqslant i \leqslant N\}$ in $C^*(A)$ with $\pi(E_i) = e_i$ for all i, where $\pi\colon C^*(A) \longrightarrow C^*(A)/\Sigma'$ is the quotient map.*

PROOF. Let $B_i \in C^*(A)$ be such that $\pi(B_i) = e_i$. We may assume that $B_i = B_i^*$. Then $B_i^2 - B_i \in \Sigma'$ for all i, so $P_n(B_i^2 - B_i) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, by changing each B_i in only finitely many coordinates, we may assume that $\|B_i^2 - B_i\| < 1/100$ for all i. Then there exist α, β, $0 < \alpha < \beta < 1$, such that $\mathrm{sp}(B_i) \cap (\alpha, \beta) = \varnothing$, and a function f continuous on $\mathrm{sp}(B_i)$ with $f(x) = 0$ for $x \leqslant \alpha$ and $f(x) = 1$ for $\beta \leqslant x$. Then $f(B_i)$ is a projection in $C^*(A)$ and $\pi(f(B_i)) = f(e_i) = e_i$. Hence we can assume that all our original B_i are projections in $C^*(A)$. Now, for $i \neq j$, $\pi(B_iB_j) = 0$, so $P_n(B_iB_j) \longrightarrow 0$ as $n \longrightarrow \infty$. By changing each B_i in only finitely many coordinates we may assume that $\|B_iB_j\| < 1$ for all i, j.

Let $E_1 \equiv B_1$. Now, by Lemma 14,

$$\begin{aligned}&\|P_n(B_1) \vee P_n(B_2) - P_n(B_1) - P_n(B_2)\| \\ &\qquad \leqslant 2\|P_n(B_1)P_n(B_2)\|(1 - \|P_n(B_1)P_n(B_2)\|)^{-1/2}.\end{aligned}$$

Hence the element of $\Sigma \oplus C^*(A_n)$ given by

$$(P_n(B_1) \vee P_n(B_2)) = (P_n(B_1) \vee P_n(B_2) - P_n(B_1) - P_n(B_2)) + B_1 + B_2$$

is in $C^*(A)$. Let $E_2 \equiv (P_n(B_1) \vee P_n(B_2)) - B_1$. Then $E_2 \in C^*(A)$, $E_1E_2 = 0$ and $\pi(E_2) = \pi(B_2) = e_2$. Likewise, if we let

$$E_3 \equiv (P_n(E_1) \vee P_n(E_2) \vee P_n(B_3)) - E_1 - E_2,$$

then $E_3 \in C^*(A)$, $\pi(E_3) = e_3$ and $\{E_1, E_2, E_3\}$ are an orthogonal family. The proof is completed by an induction argument which we omit.

The next theorem is the key to proving the structure theorem that we promised in the remarks before Lemma 13. The method used in the proof of Theorem 16 is closely related to the proofs of Lemmas 1.9 and 1.10 in [9]. The fact that matrix units in the Calkin algebra lift to "almost matrix units" (namely, $E_{ij}^* = E_{ji}$, $E_{ij}E_{kl} = \delta_{jk}E_{il}$, and ΣE_{ii} is a projection of finite codimension) was stated in the preliminary version of [1]. A more general theorem has been proved by F. J. Thayer [*Liftings in the category of C^*-algebras*, Thesis, Harvard Univ., 1972], and the result for the Calkin algebra has been proved in [16]. But we cannot use this fact, since we again need to insure that the matrix units lift to $C^*(A)$.

THEOREM 16. *Suppose $A = \Sigma_{n=1}^{\infty} \oplus A_n$, with each A_n an irreducible finite matrix and $C^*(A) \supset \Sigma'$. Further suppose that the sequence $\{P_n\colon 1 \leqslant n\}$ converges to a unique irreducible representation π with $N = \dim(H_\pi) \geqslant 2$. Let $\{e_{ij}\colon 1 \leqslant i, j \leqslant N\}$ be elements of $C^*(A)/\Sigma'$ such that $e_{ij}^* = e_{ji}$, $e_{ij}e_{lk} = \delta_{jl}e_{ik}$, $\Sigma_{i=1}^N e_{ii} = I$.*

Then there exist $E_{ij} \in C^*(A)$ such that $E_{ij}^* = E_{ji}$, $E_{ij}E_{lk} = \delta_{jl}E_{ik}$, $\pi(E_{ij}) = e_{ij}$, and $P_n(\Sigma_1^N E_i) = I_n$ for all n greater than some n_0.

PROOF. Let $B_{ij} \in C^*(A)$ be such that $\pi(B_{ii}) = e_{ij}$. By Lemma 15 we may assume that the $\{B_{ii}\colon 1 \leqslant i \leqslant N\}$ are orthogonal projections. Now for every $i \neq 1$ we have $\pi(B_{11} - B_{11}B_{i1}^*B_{i1}B_{11}) = 0$.

Hence $P_n(B_{11} - B_{11}B_{i1}^*B_{i1}B_{11}) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus by making B_{11} zero in the first few coordinates we can assume that

$$\|B_{11} - B_{11}B_{i1}^*B_{i1}B_{11}\| < 1 \quad \text{for all } i.$$

Now if C is the abelian C^*-algebra (without I) generated by B_{11} and $B_{11}B_{i1}^*B_{i1}B_{11}$ then C has B_{11} as identity and $B_{11}B_{i1}^*B_{i1}B_{11}$ is positive and invertible in C. Hence if $X_{1i} \in C \subset C^*(A)$ is the positive square root of the inverse of $B_{11}B_{i1}^*B_{i1}B_{11}$, then $B_{11} = X_{1i}^2B_{11}B_{i1}^*B_{i1}B_{11}$. We note that

$$e_{11} = \pi(X_{1i})^2 e_{11}e_{1i}e_{i1}e_{11} = \pi(X_{1i})^2 e_{11} = \pi(X_{1i})^2,$$

and hence $\pi(X_{1i}) = e_{11}$. Now let $W_{1i} = X_{1i}B_{11}B_{i1}^*$. Then $W_{1i}W_{1i}^* = X_{1i}B_{11}B_{i1}^*B_{i1}B_{11}X_{1i} = B_{11}$, also $\pi(W_{1i}) = e_{11}e_{11}e_{1i} = e_{1i}$ and $\pi(W_{1i}^*W_{1i}) =$

e_{ii}. Hence $P_n(W_{1i}^*W_{1i} - B_{ii}) \longrightarrow 0$ as $n \longrightarrow \infty$, and there is an m_i such that

$$\|P_n(W_{1i}^*W_{1i} - B_{ii})\| < 1 \quad \text{for all } n \geqslant m_i.$$

Now for $m \geqslant m_0 = \sup\{m_i\colon 2 \leqslant i \leqslant n\}$, let

$$C_{im} = P_m(I - B_{ii} - W_{1i}^*W_{1i} + B_{ii}W_{1i}^*W_{1i} + W_{1i}^*W_{1i}B_{ii}).$$

Now by [4, §4] each C_{im} is positive and invertible and if we let

$$S_{im} = C_{im}^{-1/2}P_m(W_{1i}^*W_{1i} + B_{ii} - I)$$

then S_{im} is a selfadjoint unitary and

$$S_{im}P_m(W_{1i}^*W_{1i})S_{im} = P_m(B_{ii}).$$

Let $S_i \in \Sigma \oplus C^*(A_n)$ be defined by

$$P_m(S_i) = I \quad \text{if } m < m_0$$

and

$$P_m(S_i) = S_{im} \quad \text{if } m \geqslant m_0.$$

Then, for each i, $2 \leqslant i \leqslant N$, S_i is a selfadjoint unitary. Now

$$\pi(I - B_{ii} - W_{1i}^*W_{1i} + B_{ii}W_{1i}^*W_{1i} + W_{1i}^*W_{1i}B_{ii}) = I - e_{ii} - e_{ii} + 2e_{ii} = I,$$

so $\|C_{im} - I_m\| \longrightarrow 0$ as $m \longrightarrow \infty$, and $\|C_{im}^{-1/2} - I_m\| \longrightarrow 0$ as $m \longrightarrow \infty$. Also $\pi(W_{1i}W_{1i} + B_{ii} - I) = 2e_{ii} - I$. Hence $\|S_{im} - P_m(2B_{ii} - I)\| \longrightarrow 0$ as $m \longrightarrow \infty$. Thus $(S_i - (2B_{ii} - I)) \in \Sigma' \subseteq C^*(A)$ and hence $S_i \in C^*(A)$ and $\pi(S_i) = 2e_{ii} - I$. Now change all the B_{ii}, B_{11}, B_{i1}, W_{1i}, X_{1i} to be zero in the first $m_0 - 1$ coordinates. Then all our previous equations still are true. (We do not go back and redo the proof, we just alter the operators we have.) Also

$$S_i(W_{1i}^*W_{1i})S_i = B_{ii}.$$

Let $U_{i1} = S_iW_{1i}^* \in C^*(A)$. Then $U_{i1}^*U_{i1} = B_{11}$, $U_{i1}U_{i1}^* = B_{ii}$, and $\pi(U_{i1}) = e_{i1}$. Now let $E_{ii} = B_{ii}$ for all i and let $E_{ij} = U_{i1}U_{j1}^*$ if $i \neq j$, where we let $U_{11} = B_{11}$. It then follows that $E_{ij}E_{lk} = \delta_{jl}E_{ik}$, $\pi(E_{ij}) = e_{ij}$, $E_{ij}^* = E_{ji}$. Also $\pi(\Sigma_1^N E_{ii}) = I$, so, $\|P_n(I - \Sigma_1^N E_i)\| < 1$ for all n greater than some n_0. But $P_n(I - \Sigma_1^N E_i)$ is a projection, hence $P_n(\Sigma_1^N E_i) = I_n$ for all n greater than some n_0.

We have stated and proved Theorem 16 in the form given for notational convenience, but the same proof proves a more general statement: Suppose $A = \Sigma_{n=1}^{\infty} \oplus A_n$ with each A_n an irreducible finite matrix and $C^*(A) \supset \Sigma'$. Further suppose that the sequence $\{P_n\colon 1 \leqslant n\}$ converges to a finite number of unitarily inequivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_m$, each of which is finite dimensional. Then $C^*(A)/\Sigma'$ is isomorphic to $\mathcal{B}(H_{\pi_1}) \oplus \mathcal{B}(H_{\pi_2}) \oplus \ldots \oplus \mathcal{B}(H_{\pi_m})$. If $\{e_{ij}^{(k)}: 1 \leqslant i, j \leqslant \dim H_{\pi_k}\}$ is a set of matrix units for $\mathcal{B}(H_{\pi_k})$, $1 \leqslant k \leqslant m$, then there are elements $E_{ij}^{(k)} \in C^*(A)$ such that $(E_{ij}^{(k)})^* = E_{ji}^{(k)}$, $E_{ij}^{(k)}E_{lt}^{(k)} = \delta_{jl}E_{it}^{(k)}$, $\pi(E_{ij}^{(k)}) = e_{ij}^{(k)}$, and $P_n(\Sigma E_{ii}^{(k)}) = I_n$ for all n greater than some n_0, where the summation is taken over all i and k.

We also remark that by using C. Olsen's theorem [14, Theorem 2.3] J. Calkin's original method of lifting partial isometries out of the Calkin algebra can be altered to give a more elegant proof of Theorem 16 in the case $N = 2$.

COROLLARY 17. *Let A be as in Theorem* 16. *Suppose that relative to the matrix units* (e_{ij}), $\pi(A)$ *has the matrix* $(\beta_{ij}) \cong B$. *Then there is an* n_0 *such that, for all* $n \geqslant n_0$, $A_n \cong B \otimes I + K_n$ *with* $\|K_n\| \longrightarrow 0$.

PROOF. Let $D = \Sigma\beta_{ij}E_{ij}$ where the E_{ij} are as in Theorem 16. Then $\pi(D) = \pi(A)$, so $A - D \in \Sigma'$ and $A = D + K$ with $K \in \Sigma'$. But, for $n \geqslant n_0$, the E_{ij} are matrix units for $C^*(A_n)$. Hence $D_n \cong I \otimes B \cong B \otimes I$ for $n \geqslant n_0$ and $\|K_n\| \longrightarrow 0$.

By using the remarks just after the proof of Theorem 16, a structure theorem for A similar to Corollary 17 could be stated and proved in the case that the sequence $\{P_n: 1 \leqslant n\}$ converges to a finite number of unitarily inequivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_m$, each of which is finite dimensional. In this case we would have that for large n: $A_n \cong \Sigma_{i=1}^m \bigoplus (\pi_i(A) \otimes I_i) + K_n$, where each I_i is the identity on an appropriate space.

Also, if we do not assume that the P_n are discrete in $C^*(A)\hat{}$ and assume instead that the sequence $\{P_n: 1 \leqslant n\}$ converges to P_l for some l, then by considering $A' = \Sigma' \bigoplus A_n$, where now the prime denotes the fact that l is not included in the summation, we still have that the conclusion of Corollary 17 is valid, except that the n_0 must be chosen greater than l. If $A = \Sigma \bigoplus A_n$, each A_n a finite irreducible matrix and $C^*(A)\hat{}$ Hausdorff with a finite number of cluster points for $\{P_n: 1 \leqslant n\}$, then by partitioning $\{P_n: 1 \leqslant n\}$ into a finite number of disjoint subsequences, we immediately have a structure theorem similar to Corollary 17.

In order to prove a structure theorem for arbitrary operators with Hausdorff spectrum, the only case that remains is the case when $\{P_n: 1 \leqslant n\}$ has infinitely many cluster points. This case presents many complications, and we have no results in this case.

In Theorem 16 we showed that $n \times n$ matrix units can be lifted to "almost matrix units". We conclude this paper by showing that (even 2×2) matrix units in $C^*(A)/\Sigma'$ or in $\mathcal{B}(H)/\mathcal{K}(H)$ cannot in general be lifted to matrix units in $C^*(A)$ or in $\mathcal{B}(H)$. Although this result is known (see the preliminary version of [1]) for $B(H)/K(H)$ we include a proof and comments for completeness.

LEMMA 18. *Let U_+ denote the unilateral shift of multiplicity one on H. Then the operator $T = \begin{pmatrix} 0 & U_+ \\ 0 & 0 \end{pmatrix}$ on $H \oplus H$ is not unitarily equivalent to a compact perturbation of $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$.*

PROOF. Suppose that $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ is a unitary operator implementing the unitary equivalence between

$$\begin{pmatrix} 0 & U_+ \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} K_1 & I + K_2 \\ K_3 & K_4 \end{pmatrix}$$

where $K_1, K_2, K_3, K_4 \in K(H)$. Straightforward calculations yield that $Z = U_+^*(XK_1 + YK_3)$, $K_1 = X^*U_+Z$, $I + K_2 = X^*U_+W$, $K_3 = Y^*U_+Z$, $K_4 = Y^*U_+W$, and $XX^* + YY^* = X^*X + Z^*Z = Y^*Y + W^*W = WW^* + ZZ^* = I$. Hence $Z \in K(H)$ and $Y^*U_+ = Y^*U_+WW^* + Y^*U_+ZZ^* = K_4W^* + K_3Z^*$. It then follows that $Y \in K(H)$. Since $Z, Y \in K(H)$, we have that $I - XX^*$, $I - X^*X$, $I - WW^*$ and $I - W^*W \in K(H)$. By [1, Theorem 3.1], X and W are compact perturbations of isometries or co-isometries of finite defect. In particular, they are Fredholm operators. Since the index of the product of Fredholm operators is the sum of the indices we obtain that

$$\begin{aligned} 0 = \operatorname{ind}(I + K_2) &= \operatorname{ind}(X^*) + \operatorname{ind}(U_+) + \operatorname{ind}(W) \\ &= -\operatorname{ind}(X) - 1 + \operatorname{ind}(W). \end{aligned}$$

But we also have that

$$0 = \operatorname{ind}\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \operatorname{ind}\begin{pmatrix} X & 0 \\ 0 & W \end{pmatrix} = \operatorname{ind}(X) + \operatorname{ind}(W).$$

Adding these two equations yields $1 = 2\operatorname{ind}(W)$, which is a contradiction.

One easily verifies that the operator T in Lemma 18 is unitarily equivalent to the operator

$$0 \oplus \sum_{n=1}^{\infty} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{on } \mathbf{C} \oplus \sum_{n=1}^{\infty} \oplus\, \mathbf{C}^2.$$

We remark that the conclusion of Lemma 18 holds if and only if the multiplicity of the shift U_+ is odd. The above-mentioned description of T and Lemma 18 establish that there exist operators S (for example, $S = \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$) and one dimensional operators F (for example $F = 0$) such that $S \oplus F$ is not unitarily equivalent to a compact perturbation of S.

PROPOSITION 19. *Matrix units in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ do not necessarily lift to matrix units in $\mathcal{B}(\mathcal{H})$. In fact, there exist operators A as in Theorem* 16 *such that matrix units in $C^*(A)/\Sigma'$ do not lift to matrix units in $C^*(A)$.*

PROOF. Let $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, let $A_1 = 0$ on $\mathbf{C}$, and for $n \geq 2$ let $A_n = B \otimes I_n + K_n$ be an irreducible operator on $\mathbf{C}^{2n}$ with $\|K_n\| < 1/n$. Then $A = \Sigma_{n=1}^{\infty} \bigoplus A_n$ is as in Theorem 16. Now A is a compact Σ'-perturbation of the operator $0 \oplus \Sigma_{n=2}^{\infty} \bigoplus (B \otimes I_n)$, which in turn is unitarily equivalent to the operator T of Lemma 18. Hence $\pi(A)$ generates a set of 2×2 matrix units in $C^*(A)/\Sigma'$ and hence in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. However, if A were a compact perturbation of a partial isometry V whose initial space V^*V and final space VV^* sum to I, then, since with respect to this decomposition V is of the form $\begin{pmatrix} 0 & U \\ 0 & 0 \end{pmatrix}$ where U is a unitary operator from V^*V onto VV^* which is in turn unitarily equivalent to $\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}$, we would obtain a contradiction from Lemma 18.

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